

All Colors Shortest Path Problem

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Abstract

All Colors Shortest Path problem defined on an undirected graph aims at finding a shortest, possibly non-simple, path where every color occurs at least once, assuming that each vertex in the graph is associated with a color known in advance. To the best of our knowledge, this paper is the first to define and investigate this problem. Even though the problem is computationally similar to generalized minimum spanning tree, and the generalized traveling salesman problems, allowing for non-simple paths where a node may be visited multiple times makes All Colors Shortest Path problem novel and computationally unique. In this paper we prove that All Colors Shortest Path problem is NP-hard, and does not lend itself to a constant factor approximation. We also propose several heuristic solutions for this problem based on LP-relaxation, simulated annealing, ant colony optimization, and genetic algorithm, and provide extensive simulations for a comparative analysis of them. The heuristics presented are not the standard implementations of the well known heuristic algorithms, but rather sophisticated models tailored for the problem in hand. This fact is acknowledged by the very promising results reported.

Keywords: NP-hardness, inapproximability, LP-relaxation, heuristic algorithms, simulated annealing, ant colony optimization, genetic algorithm.

1. Introduction

Given an undirected edge weighted graph where each vertex has an apriori assigned color, All Colors Shortest Path (*ACSP*) problem is defined as a generic problem in which the aim is to find a shortest possibly non-simple path that starts from a designated vertex, and visits every color at least once. As the same node might need to be visited multiple times, the path is not necessarily simple. This makes *ACSP* a novel and unique problem that has never been studied before to the best of our knowledge. As the problem is generic enough, it can be applied to a broad range of possible areas including mobile sensor roaming, path planning, and item collection.

In this paper, we study *ACSP* problem, prove that the problem is NP-hard, and that a constant factor approximation algorithm cannot exist unless $P = NP$. An ILP formulation is developed for *ACSP*, and elaborate heuristic solutions to this optimization problem are also provided. These heuristics are based on LP-relaxation, simulated annealing, ant colony optimization, and genetic algorithm.

An experimental study is carried out to compare them, and report the results.

The remainder of the paper is organized as follows. In Section 2, we discuss the related work, and position our paper with respect to the state of the art. In Section 3, we formally define the problem, and provide the intractability proof along with an inapproximability result. Section 4 presents an ILP formulation for *ACSP*. In Section 5, we discuss the heuristic solutions we propose. The experimental results are presented in Section 6, and the paper is concluded in Section 7.

2. Related Work

ACSP, defined and investigated in this paper, has actually features that make it look similar to a variety of problems studied extensively in the literature, each of which, however, has one or more discrepancies making *ACSP* computationally unique. Among these, Generalized Minimum Spanning Tree (*GMST*) problem introduced in [16] is probably the most similar to *ACSP*. Given an undirected graph partitioned into a number of disjoint clusters, *GMST* problem is defined to be the problem of finding the minimum cost spanning tree with exactly one node from every cluster. This problem has been shown to be NP-hard in [16], and some inapproximability results are presented in [17]. Integer Linear Programming (*ILP*) formulations for this problem are presented in [5], [19], and [18]. There exist formulations for also a variant of *GMST* in [4] and

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[10] where *at least one* instead of *exactly one* node from each cluster is visited. We refer to the latter version as ℓ -GMST. Even though there are such formulations, ACSP still differs in the shape of the solution. While ACSP outputs a possibly non-simple path, ℓ -GMST returns a tree. Moreover, it can be easily noted that a minimum spanning tree returned by ℓ -GMST can only give a rough estimate for the size of a possibly non-simple shortest path visiting all the colors even when ACSP is required to return to the base it starts off as shown in Figure 1. When the nodes with the same color are perceived as disjoint clusters so as to interpret this figure as an instance of ℓ -GMST, the tree spanning nodes 1 through 6 is the optimal solution to it with cost 5.

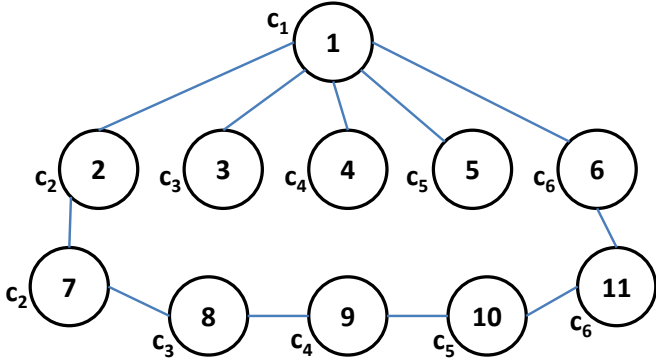


Figure 1: An example graph corresponding to an instance of ACSP. All the edges have a weight of 1, and the colors assigned to the nodes are shown next to them. Node 1 is designated as the base. The shortest path for this instance of ACSP is 1, 2, 7, 8, 9, 10, 11 which has a length of 6. When the path is constrained to return to the base, however, the path length of the solution becomes 8.

Another problem seemingly similar to ACSP is Generalized Traveling Salesman Problem (GTSP) formulated first in [12]. Given a group of possibly intersecting clusters of nodes, GTSP tries to find a shortest Hamiltonian tour with at least one (or exactly one) visit to a node from every cluster. An integer linear programming formulation for GTSP when the distance matrix is asymmetrical is given in [13]. In [14], it is shown that a given instance of GTSP can be transformed into an instance of standard TSP. In [6], GTSP is noted to be NP-hard as standard TSP is a specialization of GTSP with clusters in the form of singleton nodes. It is also surprising to note as [1] demonstrates that GTSP can be transformed into standard TSP very efficiently with the same number of nodes, but with a modified distance matrix. ACSP differs from also these variants of GTSP, in that, the nodes may be visited multiple times, and the path returned need not be a cycle.

3. The Problem Definition

ACSP is modeled as a graph problem. The input to the problem is an undirected edge weighted graph where each vertex is assigned a color known in advance. The goal

is then to find the shortest possibly non-simple path that visits every distinct color at least once in this graph. The formal definition of the problem is given as:

Definition 3.1. Given an undirected graph $G(V, E)$ with a color drawn from a set C of colors assigned to each node, and a non-negative weight associated with each edge, ACSP is the problem of finding the shortest (possibly non-simple) path starting from a designated base node $s \in V$ such that every color occurs at least once on the path.

The weights $w_{i,j}$ where $(i, j) \in E$ in G correspond to distances. We will use the words *weight*, *cost*, and *distance* interchangeably throughout the paper. The cost of a solution to an instance of ACSP is simply the length of the path returned.

ACSP can easily be shown to be NP-hard by a trivial polynomial time reduction from *Hamiltonian Path* (HP) problem which is well-known to be NP-complete [7]. Given an undirected graph $G(V, E)$, HP is defined to be the problem of deciding whether it has a Hamiltonian path, namely, a simple path that visits every node in the graph exactly once.

3.1. NP-hardness of ACSP

Given an instance of HP, it can be transformed to the corresponding instance of ACSP as follows: Let the graph in the given HP instance be denoted by $G(V, E)$. A new graph $G'(V \cup \{s\}, E \cup \{(s, v) | v \in V\})$ is obtained by adding to G a new node s , and also the edges from s to all the original nodes in G . Next, a distinct color from $C = \{c_1, c_2, \dots, c_{|V|+1}\}$ is assigned to each and every node in G' . The weights associated with all the edges in G' are finally set to one. We can now state the following lemma.

Lemma 3.2. A given instance of HP represented with $G(V, E)$ has a solution if and only if the corresponding instance of ACSP obtained through the lines of transformation just depicted has a solution with length $|V|$.

Proof. Let us first prove the only if part. When the given instance of HP has a solution, there must exist a Hamiltonian path P in G given by $v_{\pi(1)}v_{\pi(2)}\dots v_{\pi(i)}v_{\pi(i+1)}\dots v_{\pi(|V|)}$ of length $|V|-1$. As P is a Hamiltonian path, the permutation π of nodes in V is such that the edges $(v_{\pi(i)}, v_{\pi(i+1)}) \in E$ for all $i \in \{1..|V|-1\}$. If we let C , and $G'(V', E')$ denote the set of $|V|+1$ colors, and the transformed graph respectively in the corresponding instance of ACSP, it is then possible to construct the path $P' = sP$ in G' with total path length $|V|$ where $s \in V$ is designated as the base node. This is apparently the shortest path visiting all distinct colors at least once.

In order to prove the if part, let us assume that we have a shortest path of length $|V|$ that starts with node s in the corresponding instance of ACSP. Since the total number of colors that needs to be visited is $|V|+1$, each distinct color, and hence, the corresponding node occurs exactly once on this path. The removal of node s readily specifies a Hamiltonian path in G of the given HP instance. \square

The following theorem can hence be stated now.

Theorem 3.3. *ACSP is NP-hard.*

Proof. It is a direct consequence of Lemma 3.2. \square

Having learned about the NP-hardness of *ACSP*, a possible next step is to explore its approximability. With this objective in mind, our attention was drawn to ℓ -GMST problem having a similar computational structure. While ℓ -GMST looks for the minimum cost spanning tree, *ACSP* seeks out a possibly non-simple path with at least one node from every cluster provided that the nodes with the same color are interpreted as disjoint clusters. The following observation is first made to associate the optimal values of the respective solutions attained by both problems when fed with the same input. It is then used to report a result regarding the approximability of *ACSP*.

Proposition 3.4. *Let I correspond to an input identified by an undirected edge weighted graph $G(V, E)$, and a function $\kappa : V \rightarrow \{1, 2, \dots, k\}$ mapping the vertices to colors. For $1 \leq i \leq k$, $V_i = \{v \in V | \kappa(v) = i\}$ induce clearly a set of k disjoint clusters, which in turn allows for a proper interpretation of I by ℓ -GMST. Then,*

$$opt_{\ell\text{-GMST}}(I) \leq \min_{j \in V} \{opt_{ACSP}(I_j)\} < 2 * opt_{\ell\text{-GMST}}(I)$$

holds for all valid instances I , where I_j is obtained from I by designating $j \in V$ as the base node, and opt_A returns the cost of the optimal solution to its argument interpreted as an instance of either one of the two problems as dictated by the subscript A .

Proof. Let us assume that the first inequality in the proposition does not hold, and, there is an instance I for which $opt_{\ell\text{-GMST}}(I) > \min_{j \in V} \{opt_{ACSP}(I_j)\}$. Let us suppose that $s \in V$ is a node that minimizes the right-hand side of this inequality. In that case, the solution to *ACSP* with cost $opt_{ACSP}(I_s)$ can be easily reworked, by simply eliminating any cycles, and duplicate edges, into a tree T' . T' is clearly a solution to ℓ -GMST for the instance I with cost less than $opt_{\ell\text{-GMST}}(I)$, and hence, contradicting the assumption.

Let us assume now the latter inequality does not hold. This, for at least one instance of input I , leaves us with $\min_{j \in V} \{opt_{ACSP}(I_j)\} \geq 2 * opt_{\ell\text{-GMST}}(I)$. Let us also assume that the tree $T'(V', E')$ is a solution to ℓ -GMST with cost $opt_{\ell\text{-GMST}}(I)$ for instance I . Rooting T' at some $s \in V'$, a possibly non-simple path starting from s could be constructed visiting all the nodes in it by a depth-first search. This path, however, forms a solution to *ACSP* for instance I_s with cost strictly less than $2 * opt_{\ell\text{-GMST}}(I)$ as no edge gets visited more than twice, and there exists at least one edge that is visited exactly once given that the return to the base is not performed upon hitting the last leaf node in V' . This, however, contradicts the assumption. \square

It is shown in [10] that ℓ -GMST, referred to as CLASS TREE problem in the paper, does not have a constant-factor polynomial time approximation algorithm (*apx*) unless $P = NP$.

Theorem 3.5. *ACSP does not have a constant-factor polynomial time approximation algorithm unless $P = NP$.*

Proof. Let us assume, to the contrary, that *ACSP* has an *apx* denoted by apx_{ACSP} . Based on this assumption, an *apx* for ℓ -GMST can be shown to also exist, and hence a contradiction, as follows.

Given any valid input I for ℓ -GMST, consisting of an undirected graph $G(V, E)$ along with disjoint clusters $V_i \subseteq V$ with $1 \leq i \leq k$, we denote by I_j the input for *ACSP* obtained from I by designating $j \in V$ as the base. The initial assumption with regard to the existence of an *apx* suggests by definition

$$opt_{ACSP}(I_j) \leq apx_{ACSP}(I_j) \leq c * opt_{ACSP}(I_j)$$

for some constant $c > 1$, and all valid input I_j where $j \in V$. Taking the minimum over all $j \in V$, we obtain

$$\min_{j \in V} \{opt_{ACSP}(I_j)\} \leq \min_{j \in V} \{apx_{ACSP}(I_j)\} \leq c * \min_{j \in V} \{opt_{ACSP}(I_j)\}.$$

Combining this result with Proposition 3.4,

$$opt_{\ell\text{-GMST}}(I) \leq \min_{j \in V} \{apx_{ACSP}(I_j)\} < 2 * c * opt_{\ell\text{-GMST}}(I)$$

is readily obtained. It should be noted that the minimization over $apx_{ACSP}(I_j)$ involves running the constant-factor approximation for *ACSP* separately for each $j \in V$, and the total time, even though amplified by a factor of $|V|$, is still polynomial in the size of a given instance. Therefore, the last inequality implies, by definition, a $2c$ -factor *apx* for ℓ -GMST. This, however, is a contradiction, and hence, the proof. \square

4. ILP Formulation of *ACSP*

In this section, an Integer Linear Programming formulation of *ACSP* is presented. To this end, we start by making the following observation first.

Proposition 4.1. *In an optimal solution to any instance of *ACSP*, no edge can be visited more than once in any given direction.*

Proof. We assume that p is a possibly non-simple path with the shortest distance, forming a solution to a given instance of *ACSP*. Contrary to the proposition, we proceed by assuming that an edge (i, j) is traversed more than once in the direction from node i to node j . Highlighting the first two occurrences of this edge, then, the path can be represented as $p = s, x, i, j, y, i, j, z$ where s is the base, and x, y , and z are sequences of zero or more nodes with edges in between consecutive nodes. It should be noted that

neither x nor y are allowed, by the assumption, to have any occurrences of i , and j consecutively in this order. We can, in that case, construct a new path $p' = s, x, i, y^R, j, z$ with y^R corresponding, in reverse order, to the sequence of nodes in y . This new path, p' , visiting the same set of nodes as p , however, is shorter by $2 * w_{i,j}$ than p . This contradicts the optimality of p , and hence, proving the proposition. \square

Proposition 4.1 allows for an ILP formulation to *ACSP* where tracking down whether an edge is visited as part of an optimal solution in either one of the two possible directions becomes possible by employing a binary decision variable. This observation, coupled with the motivation to come up with a compact ILP model, form the basis of the transformation to be described next. At the heart of the transformation is the replacement of each undirected edge in a given instance of *ACSP* with two directed edges, and hence the adoption of a directed graph view as a substitute in the ILP formulation.

Let us assume that an instance of *ACSP* is given, as determined by an undirected edge weighted graph $G(V, E)$, the designated base vertex $s \in V$, and $\kappa : V \rightarrow C$ mapping the vertices $V = \{1, \dots, n\}$ to colors $C = \{1, \dots, k\}$. Finally, the weights associated with the edges in G are denoted by $w_{i,j}$ for all unordered pairs (i, j) (or $\{i, j\}$) $\in E$. It is therefore implicitly assumed that $w_{i,j} = w_{j,i}$ for all $(i, j) \in E$.

In transforming $G(V, E)$ to a directed graph $G'(V', E')$ to be used in the ILP formulation, we first introduce two new nodes numbered 0 as the source, and $n+1$ as the sink, setting effectively $V' = V \cup \{0, n+1\}$ in G' . Besides, the source, and the sink are both assigned to a new color 0, extending the color set to $C' = C \cup \{0\}$. With the addition of the new color, κ is also augmented accordingly with $\kappa(0) = \kappa(n+1) = 0$. Then, a directed edge $(0, s)$ from the new source to the base s , as well as directed edges $(i, n+1)$ to the sink, for all $i \in V$ in G , are added into G' with their weights set to 0. Lastly, each undirected edge $(i, j) \in E$ is replaced by two directed edges (i, j) and (j, i) in G' with both of whose weights initialized to the weight of the original edge. With this final step, the transformation sets $E' = \{(i, j), (j, i) | (i, j) \in E\} \cup \{(0, s)\} \cup \{(i, n+1) | i \in V\}$ in G' . Continuing to use the same notation for weights in G' , $w_{0,s} = 0$, and $w_{i,n+1} = 0$ for all $i \in V$ are added after the existing $w_{i,j} = w_{j,i}$ for all unordered pairs $(i, j) \in E$.

Any possibly non-simple path, p , starting from the designated base s , and visiting all colors at least once in G , corresponds precisely to the path $0, p, n+1$ in G' , where nodes 0, and $n+1$ are the source, and the sink respectively. In the same way, a possibly non-simple path $p = 0, p', n+1$ in G' , where p' is a possibly non-simple path starting at s , and with length at least one, corresponds to p' in G . As a result, the feasible solutions in G , and G' will be in one-to-one correspondence, as long as the ILP formulation of *ACSP* can place a restriction on any feasible solution in G' to start from the source, and to terminate at the

sink. Moreover, these corresponding solutions have both the same cost. It is hence obvious that a solution to an instance of *ACSP* on G as given above is optimal if and only if the corresponding solution on the transformed instance employing G' is also optimal.

The ILP formulation for a given instance of *ACSP* can now be stated with reference to the transformation described above.

$$\text{minimize} \quad \sum_{(i,j):(i,j) \in E'} x_{i,j} * w_{i,j} \quad (1)$$

subject to

$$x_{0,s} = 1 \quad (2)$$

$$\sum_{\substack{(i,j):(i,j) \in E' \\ \wedge \kappa(j)=c}} x_{i,j} \geq 1, \quad \forall c \in C' \quad (3)$$

$$\sum_{j:(j,i) \in E'} x_{j,i} = \sum_{j:(i,j) \in E'} x_{i,j}, \quad \forall i \in V \quad (4)$$

$$y_j \geq x_{i,j}, \quad \forall (i, j) \in E' \quad (5)$$

$$\sum_{i:(i,j) \in E'} x_{i,j} \geq y_j, \quad \forall j \in V' \setminus \{0\} \quad (6)$$

$$\sum_{j:(j,i) \in E'} f_{j,i} = y_i + \sum_{j:(i,j) \in E'} f_{i,j}, \quad \forall i \in V \quad (7)$$

$$x_{i,j} \leq f_{i,j} \leq (n+1) * x_{i,j}, \quad \forall (i, j) \in E' \quad (8)$$

$$x_{i,j} \in \{0, 1\}, \quad \forall (i, j) \in E' \quad (9)$$

$$y_i \in \{0, 1\}, \quad \forall i \in V' \setminus \{0\} \quad (10)$$

$$f_{i,j} \in \{0, 1, \dots, n+1\}, \quad \forall (i, j) \in E' \quad (11)$$

The objective in this formulation is to minimize the sum of the weights over all the directed edges that have been visited as shown in (1). The binary variable $x_{i,j}$ is set to 1 when the directed edge (i, j) is visited, and to 0 otherwise. It should be noted that all the edges involving the source, and the sink, introduced later in the transformation, with weight zero have no effect on the objective. Constraint (2) ensures that the edge from the source to the base is always a part of any feasible solution. Therefore, any feasible path always starts from the source, and then moves straight to the base. Constraint (3) demands for each distinct color that the number of the visited edges directed at the nodes with this same color is at least one. As a result, every distinct color gets visited at least once. As the constraint must also hold for color 0, any feasible path is guaranteed to terminate at the sink. Constraint (4) is used to make sure that the number of the visited edges that enter into any node i in G is equal to the number of the visited edges that leave it. This obviously holds for all the nodes, but the source, and the sink in G' . The main ingredient in enforcing the shape of the solution to a possibly non-simple path is this constraint. Constraints (5), and (6) establish collectively the rules associated with the variables y_j for all $j \in V' \setminus \{0\}$. The binary decision variable y_j is set to 1 if and only if node j has been visited in a feasible solution.

Constraint (5) simply asserts that visiting an edge (i, j) is an implication of visiting node j while Constraint (6) predicates the converse. Constraint (7), along with (8), is used to eliminate any possible sub-tours, and to ensure connectedness to the base. Constraint (7) employs non-negative integer valued flow variables, denoted by $f_{i,j}$ for all edges $(i, j) \in E'$. It enforces the total flow into a visited node to be equal to one greater than the total flow out of that node. In formulating this constraint, it is assumed that the source supplies a limited amount of flow to distribute to those nodes that are visited in any feasible solution. Hence, each node visited consumes a unit flow. Constraint (8) is in charge of regulating the flow values. A flow is associated with an edge if and only if that edge is part of a solution. As the flow is conserved at all the nodes in the original graph, the base node s is no exception. Coupled with the fact that each node visited consumes a unit flow, the edge $(0, s)$ should carry as many unit flows as there are nodes to visit. Excluding the source leaves us with a maximum of $n + 1$ nodes, and hence, the factor in (8). Finally, the constraints (9), (10), and (11) are the integrality constraints for the decision variables $x_{i,j}$, y_i , and $f_{i,j}$ respectively.

5. Heuristic solutions

In this section, we describe our heuristic solutions to the intractable *ACSP* problem. Section 5.1 explains several heuristic solutions based on LP-relaxation. Simulated annealing, ant colony optimization, and genetic algorithm based heuristic solutions to *ACSP* are presented later in sections 5.2, 5.3, and 5.4 respectively.

5.1. LP Relaxation

The given ILP formulation, (1) through (11), is relaxed to an LP by replacing the integrality constraints (9), (10), and (11) with

$$0 \leq x_{i,j} \leq 1, \quad \forall (i, j) \in E' \quad (9')$$

$$0 \leq y_i \leq 1, \quad \forall i \in V' \setminus \{0\} \quad (10')$$

$$0 \leq f_{i,j} \leq n + 1, \quad \forall (i, j) \in E' \quad (11')$$

respectively. Now, the decision variables can take on real values.

We propose several heuristics based on rounding the solutions to this LP relaxation. Having learned from Theorem 3.5 that a constant-factor approximation does not exist for *ACSP*, we explore strategies based on iterative rounding, rather than typical one-shot rounding.

The first heuristic, called *LP_xACSP*, after obtaining the optimal solution to the LP relaxation, finds the maximum value strictly less than one among all $x_{i,j} \in E'$. The set of all indexes for which this maximum is attained is denoted by μ (i.e., $\mu = \arg \max_{((i,j) \in E') \wedge (0 < x_{i,j} < 1)} x_{i,j}$). Next, the LP relaxation formulation at hand is augmented with the additional constraints in the form of $x_{i,j} = 1$ for all $(i, j) \in \mu$.

Finally, a subsequent call to LP for the extended formulation is issued. Hence, the job, in this subsequent call, becomes finding the shortest possibly non-simple path that fulfills not only the previous set of constraints but also passes through every additional edge explicitly dictated by the added constraints. This process is repeated until no fractional values to process are left, and hence $\mu = \emptyset$.

The other two heuristics, called *LP_fACSP*, and *LP_{f/x}ACSP* use exactly the same strategy described above except for how μ is computed before a call to LP. While *LP_fACSP* relies on the flow variables ($\mu = \arg \max_{((i,j) \in E') \wedge (0 < x_{i,j} < 1)} f_{i,j}$) in deciding which additional $x_{i,j}$ values to round before the next iteration, *LP_{f/x}ACSP* bases its decision on the ratios of $f_{i,j}/x_{i,j}$ ($\mu = \arg \max_{((i,j) \in E') \wedge (0 < x_{i,j} < 1)} \frac{f_{i,j}}{x_{i,j}}$).

5.2. Simulated Annealing

We develop another heuristic solution for *ACSP*, based on Simulated Annealing (SA) [11]. This new heuristic, called *SA-ACSP*, can be described in three primary parts:

(1) Choosing an initial random path: The general outline of the algorithm for *SA-ACSP* is given in Figure 2. The algorithm starts with a random possibly non-simple path that visits every color at least once. Such a path is constructed by randomly extending an existing path, originating from the base, until it visits every color at least once. This process is performed only once, in line 7, at the start of each iteration in the while loop in lines 6 through 29.

(2) Generating neighbors: We generate a neighbor by removing the last node in the current state, and then, adding to a random position in the path the closest node with the same color as the removed node.

(3) Selecting the best path: Starting from an initial temperature, denoted by T in the algorithm in Figure 2, the system is cooled down until a frozen state is reached, where the temperature is close to zero. Cooling down is done by decreasing the temperature slightly at each iteration as seen in line 28. The symbol R there corresponds to the cooling rate. The energy of each state is defined by the cost of the path selected at that stage. As the temperature approaches to that of a frozen state, *SA-ACSP* keeps exploring the neighbors. At each iteration, a neighbor solution is discovered in the search space, and chosen probabilistically according to Metropolis Criterion [15],

$$p(\Delta E) = e^{-\Delta E/kT},$$

where k is Boltzmann's constant, T is the temperature, and ΔE is the difference between the energies of the current, and the neighbor solutions. If the total energy decreases, the new state is assumed right away. Otherwise, the system chooses to go to the new state according to the probability that is produced by Metropolis criterion.

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1: procedure SA-ACSP
2:    $\Delta E \leftarrow 0$ ;
3:    $bestCost \leftarrow \infty$ ;
4:    $T \leftarrow SetInitialTemperature()$ ;
5:    $iterationCount \leftarrow noOfNodes * noOfColors / 5$ ;
6:   while  $T \geq freezingTemp$  do
7:      $localBestPath \leftarrow findARandomPath()$ ;
8:      $localBestCost \leftarrow costOf(localBestPath)$ ;
9:     for  $i \leftarrow 0$  to  $iterationCount$  do
10:       $nextPath \leftarrow findANeighbourPath()$ ;
11:       $nextCost \leftarrow costOf(nextPath)$ ;
12:       $\Delta E \leftarrow nextCost - localBestCost$ ;
13:      if  $\Delta E < 0$  then
14:         $localBestPath \leftarrow nextPath$ ;
15:         $localBestCost \leftarrow nextCost$ ;
16:      else
17:         $r \leftarrow Random(0, 1)$ ;
18:        if  $r < e^{-\Delta E/T}$  then
19:           $localBestPath \leftarrow nextPath$ ;
20:           $localBestCost \leftarrow nextCost$ ;
21:        end if
22:      end if
23:    end for
24:    if  $localBestCost < bestCost$  then
25:       $bestPath \leftarrow localBestPath$ ;
26:       $bestCost \leftarrow localBestCost$ ;
27:    end if
28:     $T \leftarrow T * R$ ;
29:  end while
30:  return  $bestCost$ ;
31: end procedure

```

Figure 2: SA-ACSP: Heuristic based on simulated annealing.

5.3. Ant Colony Optimization

In this section, we present the details of how Ant Colony Optimization (ACO) [2, 3] is applied to obtain ACO-ACSP, another heuristic solution, to ACSP problem.

In ACSP, each color needs to be visited at least once. Therefore, the ant colony optimization algorithm is implemented to visit multiple food types, where each food type corresponds to a distinct color. In other words, when an ant leaves the nest, its search is not over until it finds a path that passes over every food type. The base node is chosen as the nest of all the ants, and at each iteration, the entire colony of ants is released from this nest to the graph.

The random movements of the ants, while visiting a node, are governed by an edge selection procedure. Depending on whether there is any trace of pheromone on an incident edge, the ants compute two types of edge selection probabilities. In the first one, when there is no pheromone on any incident edge, the ants make the selection based on the edge costs (or distances) using the following formula,

$$prob_{i,j} = \frac{c_0 - w_{i,j}}{\sum_{k:(i,k) \in E} (c_0 - w_{i,k})},$$

where $prob_{i,j}$ is the selection probability of edge $(i,j) \in E$, and c_0 is a constant. The second case occurs when the

pheromone level on at least one incident edge is not zero. In this case, the edge selection probability calculation is performed, based on the pheromone levels, as:

$$prob_{i,j} = \frac{(D_{i,j})^\beta * \sum_{k \in C} (Ph_{i,j}(k))^\alpha}{\sum_{q:(i,q) \in E} [(D_{i,q})^\beta * \sum_{k \in C} (Ph_{i,q}(k))^\alpha]},$$

where $Ph_{i,j}(k)$ is the pheromone level on edge $(i,j) \in E$ associated with color $k \in C$, α , and β are user defined parameters with $0 \leq \alpha \leq \beta \leq 1$, and the desirability $D_{i,j}$ of edge $(i,j) \in E$ is defined to be inversely proportional to the edge's cost as $D_{i,j} = 1/w_{i,j}$.

Pheromone level updates are carried out in two different ways, namely, the local, and the global updates. The local updates are applied to all the edges selected because each ant secretes pheromone as it moves on the edges. Moreover, the pheromones are not stored only on the edges. Ants also have some pheromones within themselves, and their levels drop while they are secreted by the ants during their traversal. Therefore, the local updates are performed on the edges as well as the ants selecting them. While the local update to the pheromone level corresponding to color $k \in C$, after the selection of edge $(i,j) \in E$ by ant t , is performed by

$$Ph_{i,j}(k) = (1 - \delta) * Ph_{i,j}(k) + \delta * Ph_t(k),$$

the level of the pheromone associated with color k stored on ant t becomes the subject of the local update

$$Ph_t(k) = Ph_t(k) - \delta * Ph_t(k),$$

where δ is a user defined evaporation parameter such that $0 \leq \delta \leq 1$. It should be noted here that the same notation has been employed to keep track of the pheromone levels on both the edges, and the ants. However, the levels of the pheromones stored on ants are tracked with a single subscripted index as opposed to two for the edges.

The second type of the update to pheromone levels comes under the title of the global update. The global pheromone update is also known as off-line pheromone update. It is applied, at the end of each iteration, only to the edges that are on the best path found so far. The pheromone level for each color $k \in C$ on each such edge is updated using the following formula

$$Ph_{i,j}(k) = (1 - \delta) * Ph_{i,j}(k) + \delta * \frac{1}{cost(bestPath)}$$

where $cost(bestPath)$ denotes the total cost of traversing the edges associated with the best, possibly non-simple, path found so far.

The pseudo-code of ACO-ACSP heuristic algorithm, reflecting the general anatomy of ant colony optimization as applied to ACSP is presented in Figure 3.

5.4. Genetic Algorithm

The Genetic Algorithm (GA) [9] has five main steps: initialization, fitness, selection, crossover, and mutation.

```

1: procedure SELECTEDGE(ANT ANT)
2:   if ant.isDone OR ant.isDiscarded then
3:     ant.pheromoneUpdate  $\leftarrow$  false;
4:   else
5:     incidentEdges  $\leftarrow$  findAvailableEdges(ant);
6:     if sizeOf(incidentEdges) = 0 then
7:       ant.isDiscarded  $\leftarrow$  true;
8:     else
9:       ant.pheromoneUpdate  $\leftarrow$  true;
10:      sum  $\leftarrow$  calcProbUsingPheromones();
11:      if sum = 0 then
12:        calcProbUsingDistances();
13:      end if
14:      updateAntToPickEdge();
15:    end if
16:  end if
17: end procedure

1: procedure ACO-ACSP()
2:   bestAnt  $\leftarrow$   $\emptyset$ ;
3:   for j  $\leftarrow$  0 to iterationCount do
4:     colony = createAnts(colonySize);
5:     antsDoneTour  $\leftarrow$  0;
6:     while antsDoneTour < colonySize do
7:       for all ant in colony do
8:         selectEdge(ant);
9:         updateLastSelectedEdge();
10:        if ant.hasAllColors then
11:          ant.isDone  $\leftarrow$  true;
12:          ++antsDoneTour;
13:        end if
14:      end for
15:    end while
16:    tempAnt  $\leftarrow$  findBestAnt();
17:    updatePheromoneOnBestPath(tempAnt.path);
18:    if tempAnt.cost  $\leq$  bestAnt.cost then
19:      bestAnt  $\leftarrow$  tempAnt;
20:    end if
21:  end for
22:  return bestAnt.cost;
23: end procedure

```

Figure 3: *ACO-ACSP*: Heuristic based on ant colony optimization.

In this section, we develop *GA-ACSP* which is another heuristic solution to *ACSP* based on GA. The algorithm is presented in Figure 4.

During the initialization step, a pool of chromosomes, called population, is generated. We encode the chromosomes in such a way that each chromosome is represented by an ordered list of vertices, corresponding to a solution to a given *ACSP* instance. As the path is not necessarily a simple path, each vertex may appear multiple times on a chromosome. Initially, the population is filled with a certain number of randomly created, possibly non-simple, paths, each of which visits all the distinct colors at least once.

In the next step, the fitness values are calculated for each chromosome in the population. The fitness value of a chromosome, in *GA-ACSP*, is simply taken as the total

```

1: procedure GA-ACSP
2:   population  $\leftarrow$   $\emptyset$ ;
3:   for i  $\leftarrow$  0 to populationSize do
4:     chr = createRandomChromosome();
5:     ensureConnectivity(chr);
6:     population.add(chr);
7:   end for
8:   for i  $\leftarrow$  0 to iterationCount do
9:     candidates  $\leftarrow$  rouletteWheelSelection(population);
10:    children  $\leftarrow$  crossOver(candidates);
11:    for all child in children do
12:      r  $\leftarrow$  random(0, 1);
13:      if r < mutationProbability then
14:        mutate(child);
15:      end if
16:      completeMissingColors(child);
17:      ensureConnectivity(child);
18:      population.add(child);
19:    end for
20:    w2c  $\leftarrow$  find2chromosomesWithLowestFitness();
21:    population.remove(w2c);
22:  end for
23:  return costOfBestChromosome;
24: end procedure

```

Figure 4: *GA-ACSP*: Heuristic based on genetic algorithm.

distance traveled down the corresponding possibly non-simple path.

In the selection step, two candidate chromosomes are selected from the population for crossover. The selection of the candidates are performed using the roulette wheel selection algorithm [8], in which the chromosomes with higher fitness values have higher chances to be selected.

In performing a crossover, two random positions p_1 , and p_2 with a common vertex are initially figured out in the first, and the second candidate chromosomes respectively. The portions beyond p_1 , and p_2 are then swapped between the candidates to produce two new children. In case the candidates do not have a common vertex, two more candidates for crossover are selected until a vertex common to both can be found. In the end, two new, possibly non simple, paths are generated. It should be noted, however, that these new paths are not guaranteed to visit all the colors. Therefore, as soon as the crossover, and the mutation steps are over, we examine the paths to find a list of the missing colors on each, and then, modify the paths accordingly so that when the process is over, each of the two new chromosomes represents also a non-simple path that visits each color at least once. In modifying the paths, we follow a greedy policy, and append to the tail of the chromosome, at each iteration, the shortest path from the tail to the closest vertex with a color not visited yet. This helps keeping the path lengths as short as possible.

The last step in *GA-ACSP* is the mutation. It is carried out by simply replacing two random vertices in the chromosome, and rearranging the path to ensure that it remains connected. In order to connect non-neighbor ver-

tices in the chromosome, the shortest path between those vertices is inserted into the chromosome.

After the crossover, and the mutation steps, the child chromosomes need to be verified to correctly represent a possibly non-simple path. After any possible problems regarding missing colors, and disconnectivity are dealt with, as described above, there is still some more work to do. First, the redundant segment at the tail part of the chromosome should be cropped if all the colors have already been seen before the beginning of that segment. Lastly, the property that no edge can be visited more than once in any direction in an optimal solution can be violated as a result of the crossover, and the mutation operations. In such a case, the property should be restored, as highlighted in the proof to Proposition 4.1.

Once the two newly formed child chromosomes are added to the population, the two chromosomes that have the worst fitness values are removed from the population.

6. Experiments

In this section, we present the results of our experiments for the proposed heuristic algorithms. We refer to LP_xACSP , LP_fACSP , and $LP_{f/x}ACSP$ under the heading of LP relaxation based algorithms whereas $SA-ACSP$, $ACO-ACSP$, and $GA-ACSP$ are treated under the category of metaheuristic algorithms. We implemented the metaheuristic algorithms in C++, and used CPLEX for ILP, and LP relaxation based heuristics. All tests are performed on computers that have AMD Phenom(tm) II X4 810 2.67 GHz CPU, and 2 Gb 400 MHz DDR2 RAM running on the 32-bit operating system Ubuntu 10.04. We conducted the experiments on randomly generated graphs with varying number of nodes, and colors as listed in Table 1, with an average node degree of 6, uniform color distribution, and an average edge weight of 10. The simulations are conducted 10 times for each graph type, and only the average, and the minimum cost values are reported.

Graph Name	Number of Nodes	Number of Colors
n50-c10	50	10
n50-c20	50	20
n50-c25	50	25
n100-c25	100	25
n100-c40	100	40
n100-c50	100	50
n200-c50	200	50
n200-c75	200	75

Table 1: The number of nodes, and colors for the randomly generated graph types used in the experiments.

In the sections to follow, the experimental results are reported, first, separately for each metaheuristic algorithm. In each of these sections, we conduct various experiments for parameter tuning, namely, to find the optimal values of

individual parameters specific to a metaheuristic. Finally, in Section 6.4, an overall comparison is presented to assess the relative performance of all the heuristics proposed, using the fine-tuned parameters.

6.1. SA-ACSP: Parameter Tuning for SA

In SA, the two parameters essential to performance are the best cooling rate with respect to time and cost, and the best temperature to be used with the best cooling rate. All the other parameters for $SA-ACSP$ are kept unchanged during these tests.

The cooling rate is used to determine the amount of decrease in the temperature value at each iteration. We tested $SA-ACSP$ with various cooling rate values as presented in Figures 5, 6, and 7. The temperature value for these tests is set at 100. Looking at Figures 5, and 6, we can observe that the cost slightly decreases with the increasing values of cooling rate parameter. In Figure 7, however, we also observe that the time increases dramatically for the cooling rate values larger than 0.999. Therefore, based on the results of these experiments, we selected the best cooling rate parameter to be 0.999.

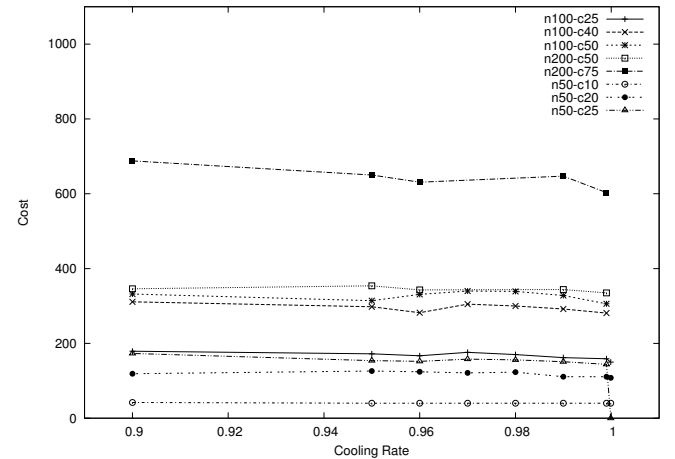


Figure 5: Minimum cost for various cooling rate values in $SA-ACSP$.

The temperature value in $SA-ACSP$ controls the probability of choosing worse paths in order to not get stuck at a local minimum. We conducted simulations with various temperature values on all graph types, and present the cost, and the CPU times in Figures 8, 9, and 10. Based on the results in these figures, we selected 1000 as the best temperature value.

6.2. ACO-ACSP: Parameter Tuning for ACO

The behavior of $ACO-ACSP$ depends on four separate parameters. In order to find the optimal value for each parameter, we conducted a series of experiments for each individual parameter. In each experiment, all the other parameters are kept constant, and only the specified parameters are tested for various values, and the cost, and

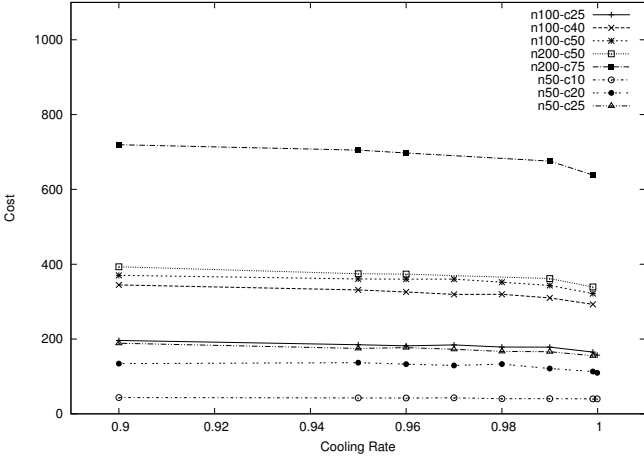


Figure 6: Average cost for various cooling rate values in *SA-ACSP*.

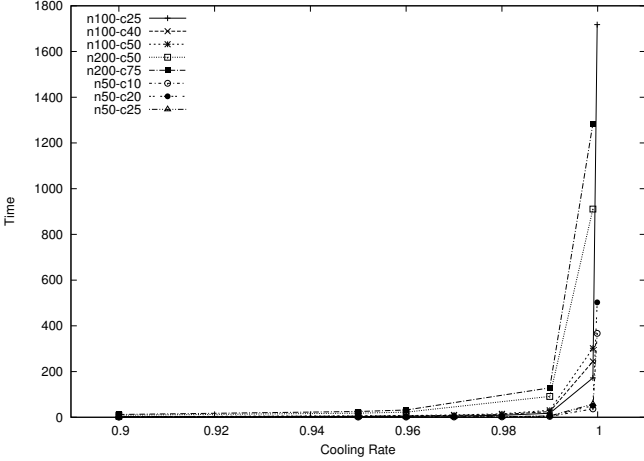


Figure 7: CPU time for calculating the average cost values for various cooling rate values in *SA-ACSP*.

the runtime values observed are recorded. The experiments conducted can be categorized into alpha-beta tests, colony size tests, and probability tests.

Alpha, and beta values are the two parameters used for edge selection in *ACO-ACSP*. The parameter alpha denotes the importance of the pheromone levels on the edges while calculating probabilities. Beta, on the other hand, represents the importance of the edge weights. The experiments are designed to decide on the combination of alpha, and beta values that gives us the best result in terms of the cost, and the CPU time. The results of the experiments are presented in Figures 11, 12, and 13. These figures report the results with respect to the CPU time, the average cost, and the minimum cost, and based on the results, we selected the alpha value as 0.4, and beta value as 0.5.

In *ACO-ACSP*, the colony size represents the number of active ants deployed at each iteration of the algorithm. Having a larger colony increases the chances for finding solutions closer to the optimal, however, at the cost of

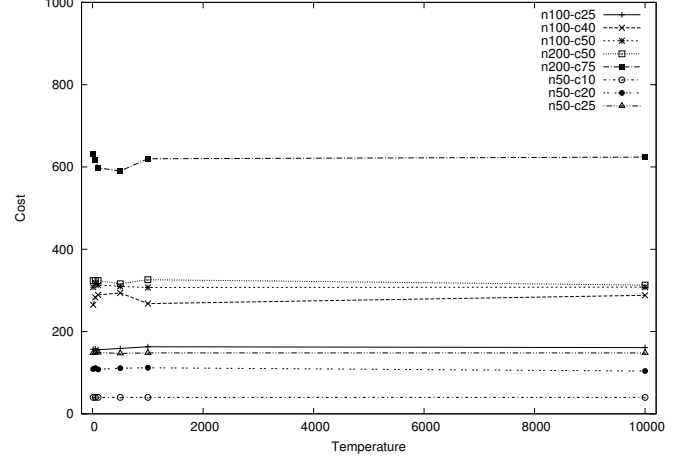


Figure 8: Minimum cost for various temperature values in *SA-ACSP*.

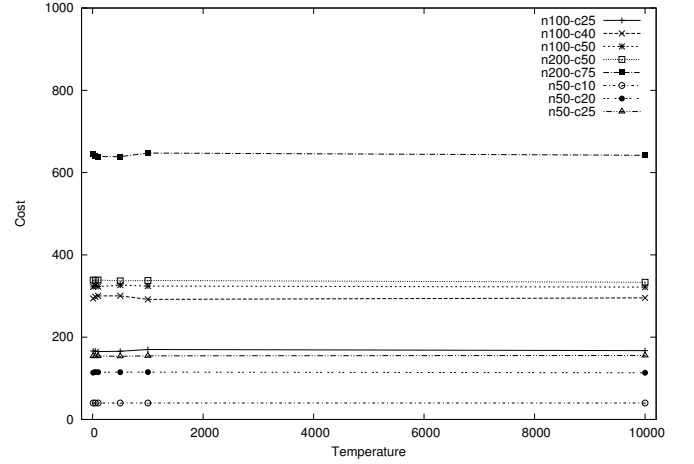


Figure 9: Average cost for various temperature values in *SA-ACSP*.

increasing the overall runtime. Therefore, we test *ACO-ACSP* for various colony sizes to decide on the optimal colony size that can achieve a minimal cost solution in an acceptable time period. The results of the experiments are presented in Figures 14, 15, and 16. As we can clearly see from the figures, the runtime increases linearly in the size of the colony, and the cost decreases only slightly for colony sizes larger than 200. Based on these results, we selected the colony size as 200 in the rest of the experiments.

The edge selection probabilities in *ACO-ACSP* are calculated based on either the level of pheromones on the edge or the edge weight itself. The results of the experiments conducted to find the optimal probability value is presented in Figures 17, 18, and 19. A close inspection of these figures reveal that both the cost, and the runtime increase for probability values larger than 0.95. Based on this observation, therefore, we selected the probability value as 0.9 to be used throughout the rest of the experiments.

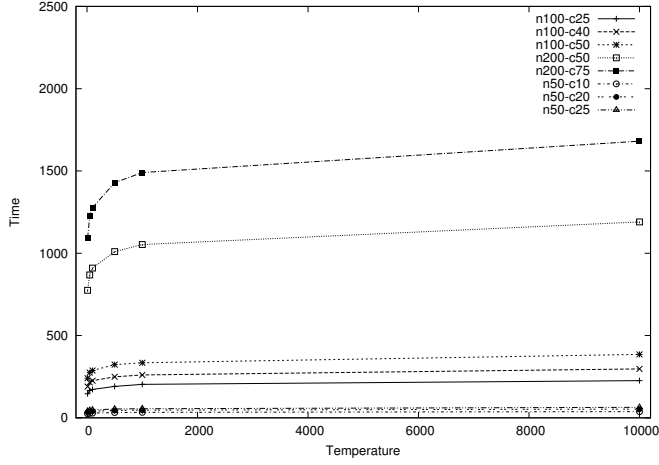


Figure 10: CPU time for calculating average cost for various temperature values in *SA-ACSP*.

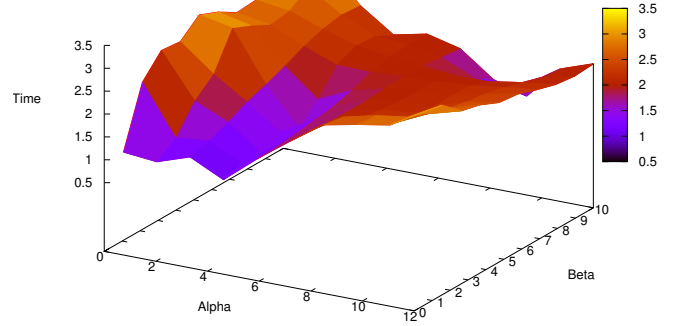


Figure 13: CPU time for calculating the average cost for various combinations of alpha, and beta values in *ACO-ACSP*.

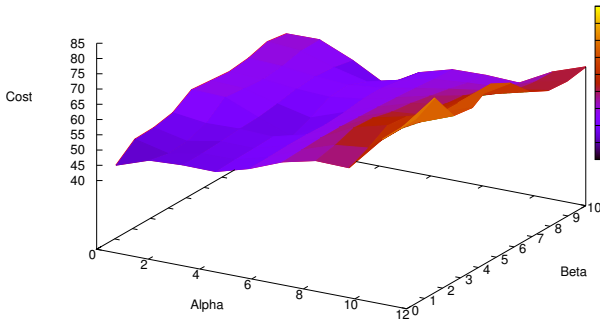


Figure 11: Minimum cost for various combinations of alpha, and beta values in *ACO-ACSP*.

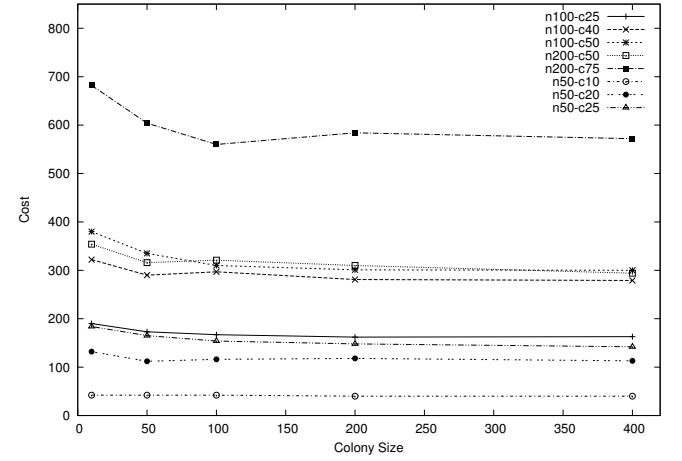


Figure 14: Minimum cost for various colony size values in *ACO-ACSP*.

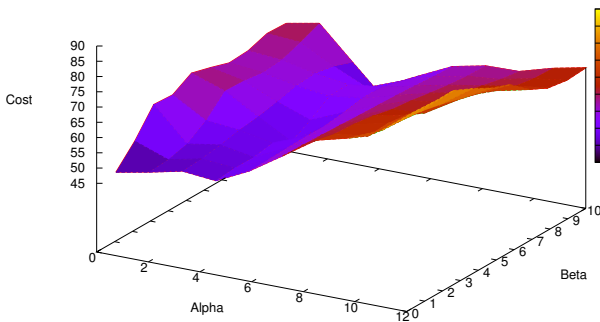


Figure 12: Average cost for various combinations of alpha, and beta values in *ACO-ACSP*.

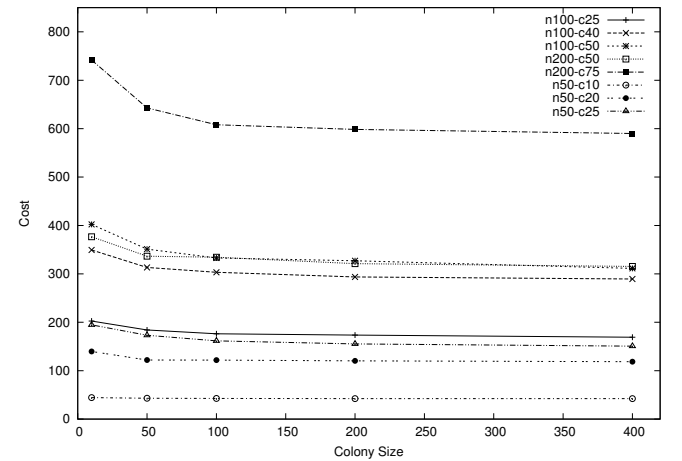


Figure 15: Average cost for various colony size values in *ACO-ACSP*.

6.3. GA-ACSP: Parameter Tuning for GA

In *GA-ACSP*, the parameters investigated are the mutation probability, the population size, and the iteration

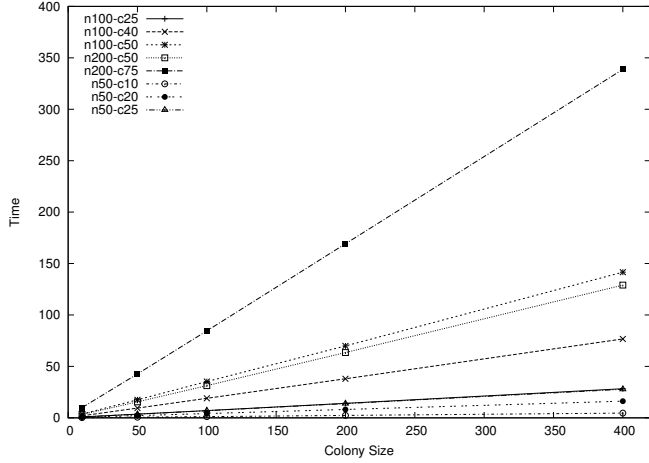


Figure 16: CPU time for calculating the average cost for various colony sizes in *ACO-ACSP*.

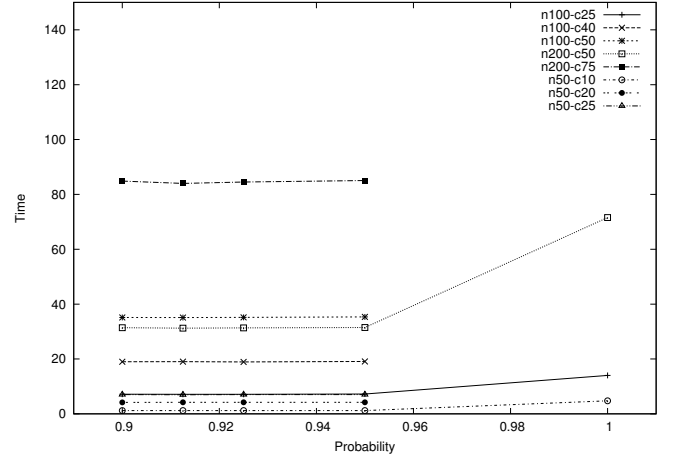


Figure 19: CPU time for calculating the average cost for various edge selection probability values in *ACO-ACSP*.

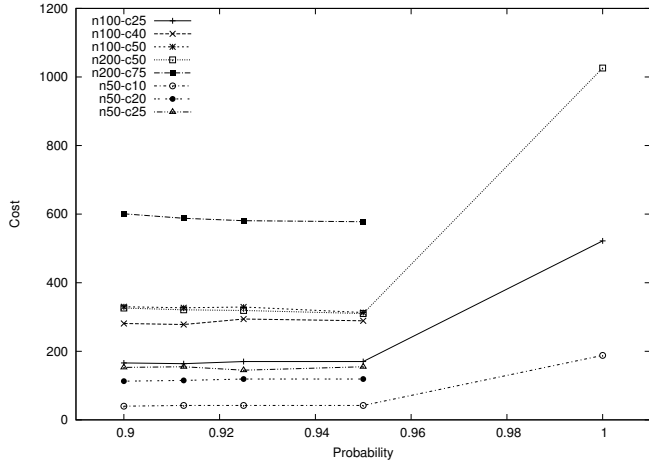


Figure 17: Minimum cost for various edge selection probability values in *ACO-ACSP*.

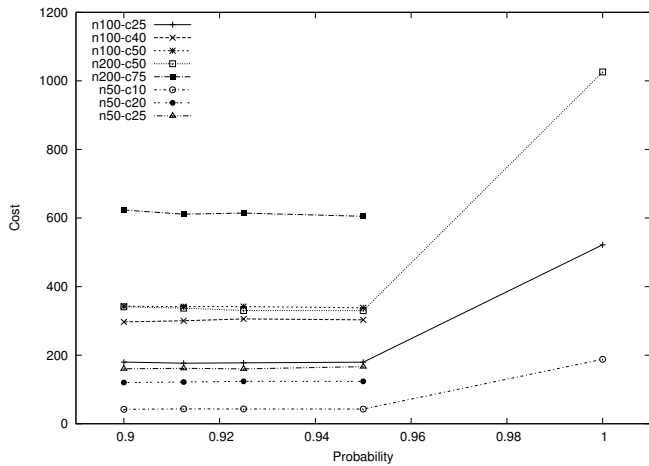


Figure 18: Average cost for various edge selection probability values in *ACO-ACSP*.

are presented in Figures 20, 21, and 22. It can be seen from the figures that the changes on the mutation probability do not affect the cost dramatically. Therefore, to prevent over-randomization of chromosomes, we selected for the mutation probability a value as low as 0.1.

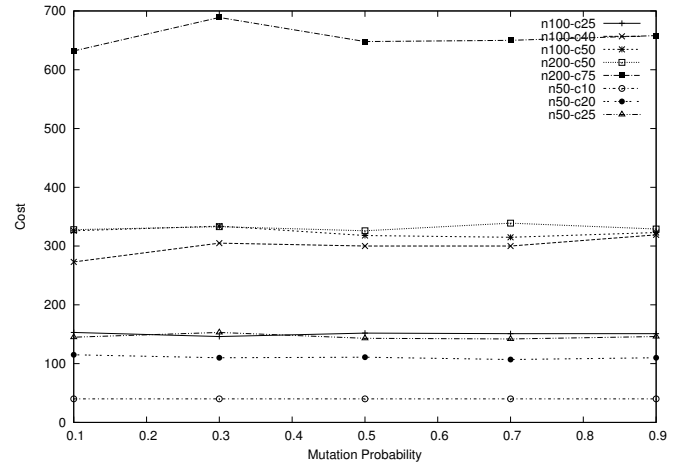


Figure 20: Minimum cost for various mutation probability values in *GA-ACSP*.

We also conducted experiments to find the optimal iteration size. The results of the experiments are presented in Figures 23, 24, and 25. Based on the results, we decided to select the iteration size as 6000, as it provides the best trade-off between the cost, and the runtime.

We also experimented on various population sizes to find the best population size for *GA-ACSP*. The results of the experiments are presented in Figures 26, 27, and 28. Based on the results, the population size is selected as 600 as both the running time, and the cost at this value of population size are lower than they are at larger population sizes.

size. The results of the tests on the mutation probability

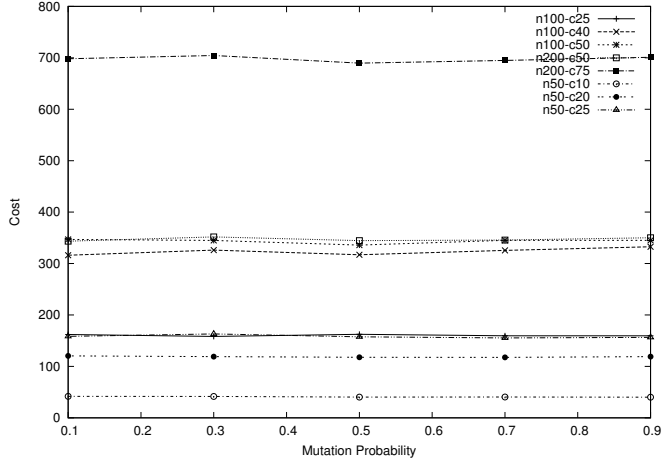


Figure 21: Average cost for various mutation probability values in *GA-ACSP*.

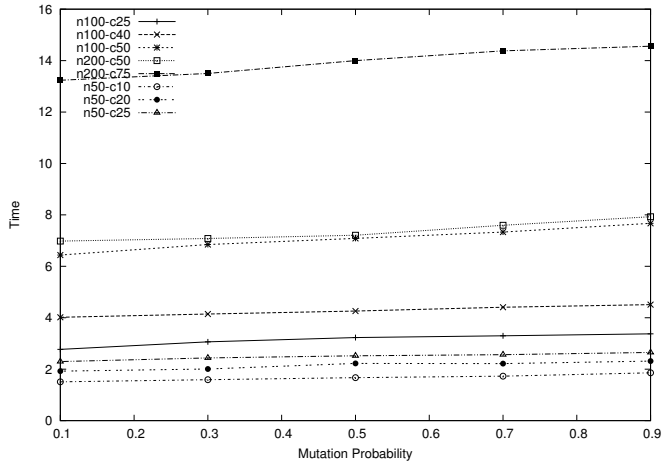


Figure 22: CPU time for calculating the average cost for various mutation probability values in *GA-ACSP*.

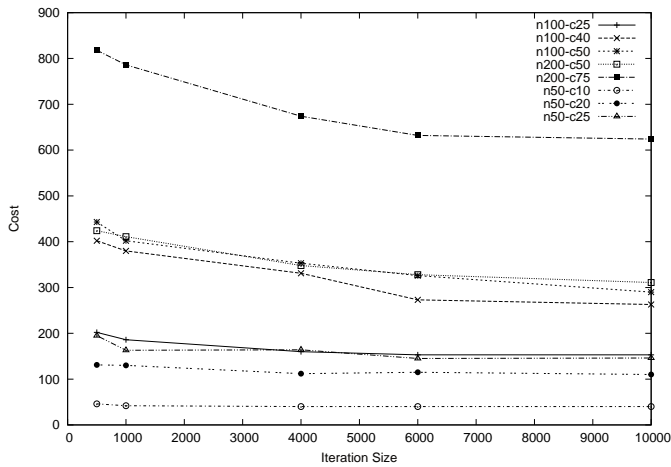


Figure 23: Minimum cost for various iteration sizes in *GA-ACSP*.

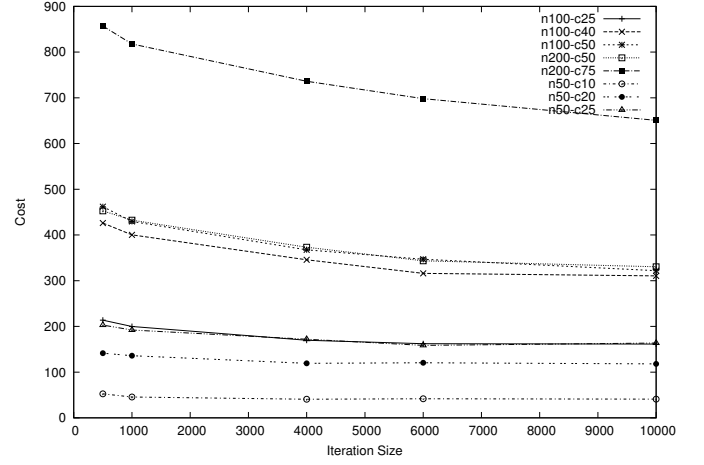


Figure 24: Average cost for various iteration sizes in *GA-ACSP*.

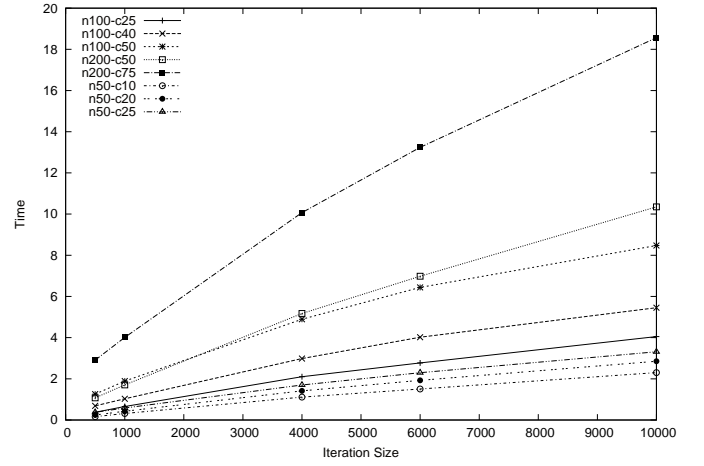


Figure 25: CPU time for calculating the average cost for various iteration sizes in *GA-ACSP*.

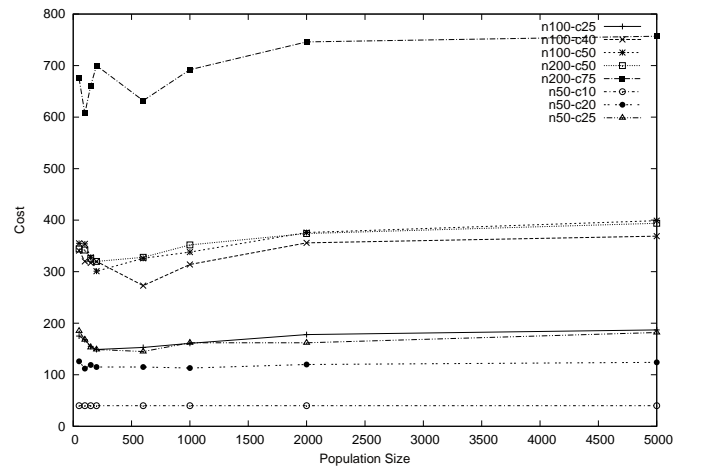


Figure 26: Minimum cost for various population sizes in *GA-ACSP*.

6.4. Comparing the Heuristic Algorithms

In this section, we first present the performance of the LP relaxation based heuristics, namely LP_xACSP ,

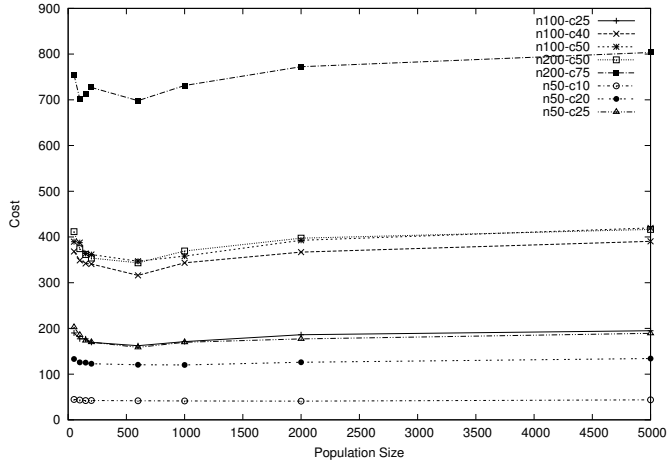


Figure 27: Average cost for various population sizes in *GA-ACSP*.

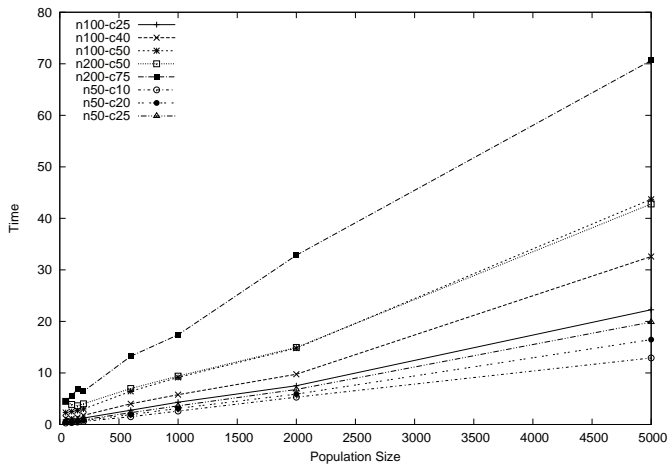


Figure 28: CPU time for calculating the average cost for various population sizes in *GA-ACSP*.

LP_fACSP , and $LP_{f/x}ACSP$ in Figure 29. The results are reported in proportion to the optimal values obtained via the ILP formulation. Next, we compare the performance of the metaheuristic algorithms, *SA-ACSP*, *ACO-ACSP*, and *GA-ACSP*. The results for them are presented, again in proportion to the optimal values, in Figure 30. For each individual metaheuristic algorithm, in these tests, the best parameter values discovered are used. We use randomly generated graphs for the types presented in Table 1.

Based on the experimental results, it is observed that the total path length returned by *SA-ACSP* is better than that returned by *ACO-ACSP* for medium-sized graphs. In contrast, *ACO-ACSP* finds lower cost paths compared to *SA-ACSP* for larger graphs. The performance of *GA-ACSP*, in terms of solution quality, is similar to the other two metaheuristics. It has, however, a remarkable advantage in terms of time spent over the other two on all types of graphs.

7. Conclusion

In this paper, a novel, and generic problem, All Colors Shortest Path (*ACSP*) problem, has been formulated, and computationally explored. *ACSP* has been shown to be NP-hard, and also inapproximable within a constant factor of the optimal. An ILP formulation has been developed for *ACSP*. Various heuristic solutions have then been devised, based on iterative rounding applied to an LP relaxation of the ILP formulation. Moreover, three different metaheuristic solutions based on simulated annealing, ant colony optimization, and genetic algorithm have been proposed. Through extensive simulations, an experimental evaluation of all the heuristics have also been reported.

The study of the computational characteristics of *ACSP* when the underlying graph is restricted to be a tree is a future work. Investigation of an approximation bound is left as an interesting open problem.

8. Acknowledgement

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Graph Name	ILP	LP_xACSP		LP_fACSP		$LP_{f/x}ACSP$	
		x	Time	f	Time	f/x	Time
n50-c10	40	3.15	1.763	1.15	1.03	2.075	0.967
n50-c20	101	1.8614	2.543	1.1584	1.357	1.1881	1.201
n50-c25	132	1.9015	2.761	1.3182	1.794	1.3636	1.186
n100-c25	138	2.1449	5.803	1.2101	3.136	1.5145	3.37
n100-c40	220	1.9955	8.986	1.3227	4.181	1.4591	4.056
n100-c50	233	1.8112	8.686	1.1631	4.014	1.1202	3.526
n200-c50	223	2.3498	40.872	1.3946	23.452	1.6906	23.556
n200-c75	399	1.9599	51.767	1.3258	30.716	1.4837	23.166

Figure 29: Comparison of ILP , LP_xACSP , LP_fACSP , and $LP_{f/x}ACSP$.

Graph Name	$SA-ACSP$			$ACO-ACSP$			$GA-ACSP$		
	Average Cost	Minimum Cost	Average Time	Average Cost	Minimum Cost	Average Time	Average Cost	Minimum Cost	Average Time
n50-c10	1	1	33.4153	1.0575	1	2.32152	1.0725	1.05	0.422245
n50-c20	1.1386	1.1089	47.0029	1.1911	1.1683	8.15708	1.2267	1.1188	0.546315
n50-c25	1.1720	1.1212	55.6508	1.1758	1.1212	13.8652	1.3303	1.2197	0.716446
n100-c25	1.2319	1.1812	203.28	1.2572	1.1739	14.0686	1.2167	1.1521	0.957005
n100-c40	1.3264	1.2182	260.757	1.3345	1.2773	38.0068	1.5386	1.4045	1.30289
n100-c50	1.3914	1.3176	334.878	1.4039	1.2918	69.8665	1.5588	1.4464	2.30258
n200-c50	1.5143	1.4619	1052.87	1.4404	1.3901	63.4437	1.5771	1.4484	2.9976
n200-c75	1.62314	1.5539	1490.63	1.5	1.4637	169.171	1.7170	1.5840	5.69891

Figure 30: Comparison of $SA-ACSP$, $ACO-ACSP$, and $GA-ACSP$ with their best parameters.

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